Lectures 13 and 14 covered sampling. Here are some practice problems for review.

**A quick tip**: When handling sampling problems, draw the spectrum. Very rarely should you have to resort to writing horrendous sums of sincs, or trying to draw the corresponding overlapping tentacle mess.

### Some comments on normalized frequency

Suppose a signal $x(t)$ is sampled at frequency $\omega_s$ to give a vector of samples $x_s[n]$. How is $\Omega$ in the DTFT $X_s(e^{j\Omega})$ of $x_s[n]$ related to $\omega$ in $X(j\omega)$, the CTFT of $x(t)$? (Don't be fooled by the notation $X_s(e^{j\omega})$ sometimes found in the lecture notes—the $\omega$ here does not correspond to the $\omega$ in $X(j\omega)$! Which is why I'll use $\Omega$ in the discussion following.)

If you are given an arbitrary vector of values such as $x_s[n]$ and told that they are consecutive samples from a signal $x(t)$ without any further information, you can't in general say anything about how $x_s[n]$ relates to $x(t)$. Is $x_s[0] = x(0)$? Is $x_s[1] = x(1)$? Or is $x_s[1] = x(0.01)$? If $x_s[0] = x(0)$ and $x_s[1] = x(1)$, does $x_s[2] = x(2)$? (Not necessarily, if you didn't sample the signal uniformly!) In other words, a vector $x_s[n]$ is just a list—a representation of some information collected from the real world. To know what it corresponds to exactly, you need more context.

So let's add a bit more context and assume $x_s[n]$ was uniformly sampled from $x(t)$—that is, $x_s[n] = x(nT)$ for some fixed period $T$. If you want to reconstruct $x(t)$ from $x_s[n]$, then you need to know $T$. Yet $x_s[n]$ itself is just a list of numbers. Even if you don't know $T$, you can still perform all kinds of analyses on $x_s[n]$ (some of which will be informative about $x(t)$). You can, for example, take the DTFT of $x_s[n]$:

$$X_s(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_s[n]e^{-j\Omega n}.$$

What does $\Omega$ actually represent here? If you take the CTFT of $x(t)$, it seems reasonable that $\omega$ ought to be a physically meaningful quantity (angular frequency in rad/sec, assuming $t$ is in seconds). But in the absence of further information, $\Omega$ itself doesn't correspond to a physical frequency, because we took the DTFT without any knowledge about how $x_s[n]$ corresponds to time in the real world. Transforming $x_s[n]$ to a different basis doesn't generate any new information; it just represents information we already have in a different form. It follows that $\Omega$ alone doesn't "know" about $T$, so it isn't a physical frequency. Instead, it's a *normalized* frequency, which is related to a physical frequency through $\Omega = \omega T$. What does this mean?

In Lecture 8 you learned that $X_s(e^{j\Omega})$ is periodic with period $2\pi$. With that understanding, we can restrict ourselves to looking at $\Omega$ from $-\pi$ to $\pi$. Let $\omega_s = \frac{2\pi}{T}$ be the sampling frequency corresponding to $T$. Then when $\Omega = \pi$, $\omega = \frac{\pi}{T} = \frac{1}{2}\omega_s$, or $\omega_s = 2\omega$. This means $\Omega = \pi$ corresponds to the *positive bandwidth of a signal that could be reconstructed without aliasing by sinc interpolation for sampling rate $T$*. In other words, $\Omega$ is what you get when you normalize a physical frequency to the sampling rate: $\Omega = \omega T = \omega \frac{2\pi}{\omega_s} = 2\pi \frac{\omega}{\omega_s}$. Pretty nifty!

**Problem 1 (Nyquist frequency)** Can the following signals be reconstructed with sinc interpolation? If so, what is the minimum frequency $\omega_s$ at which the signal should be sampled?

(a) $x(t) = \cos(2\pi t) + \sin(\pi t)$ Among these two sinusoids, the higher frequency is $2\pi$, so the signal should be sampled at some frequency $\omega_s > 2(2\pi) = 4\pi$.

(b) $x(t) = \text{rect}(t)$ This is a timelimited signal, so its Fourier transform is not bandlimited, meaning it cannot be perfectly reconstructed with sinc interpolation.

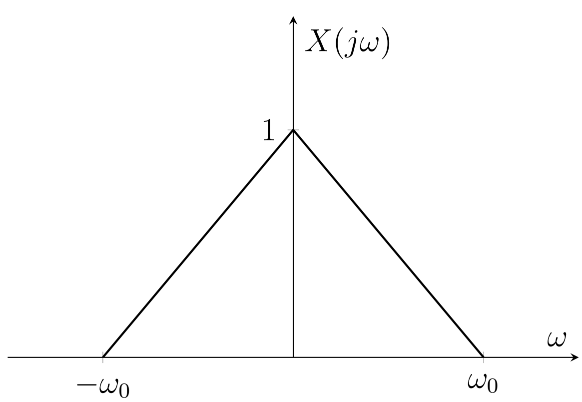
(c) $x(t) = \text{sinc}(t)$ This is a bandlimited signal with a CTFT that is the rectangular pulse of width $2\pi$. The highest frequency component is $\pi$, so the corresponding Nyquist frequency is $\omega_s = 2\pi$.

(d) $x(t) = \text{sinc}(t) * \text{sinc}(t)$ Since convolution in the time domain is multiplication in the frequency domain, this is a bandlimited signal with the same bandwidth as the signal in part (c). Thus, the corresponding Nyquist frequency is $\omega_s = 2\pi$.

(e) $x(t) = \text{sinc}^2(t)$ Since multiplication in the time domain is convolution in the frequency domain, this is a bandlimited signal with twice the bandwidth as the signal in part (c). Thus, the corresponding Nyquist frequency is $\omega_s = 4\pi$.

(f) $x(t) = e^{-t^2}$ This is not a bandlimited signal (its Fourier transform is also a Gaussian), so it cannot be perfectly reconstructed with sinc interpolation.

**Problem 2 (Aliasing)** A real analog signal $x(t)$ has CTFT $X(j\omega)$:



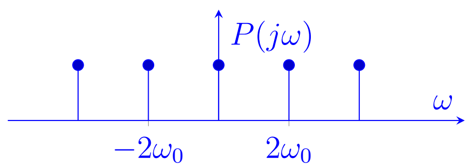

We define the impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

for sampling rate $T$.

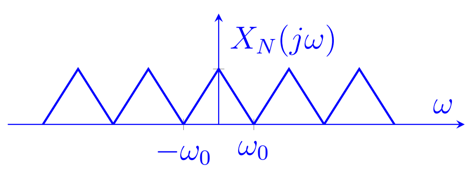
(a) Let $\omega_N = \frac{2\pi}{T_N}$ be the Nyquist frequency corresponding to a signal with the bandwidth of $x(t)$ above. What is $T_N$? Sketch $P(j\omega)$, the CTFT of $p(t)$, when $T = T_N$.

The highest frequency component is $\omega_0$, so the Nyquist frequency is $\omega_N = 2\omega_0$. Thus, $T_N = \frac{2\pi}{\omega_N} = \frac{\pi}{\omega_0}$. The impulses are spaced by $T_N$ in the time domain and by $\omega_N$ in the frequency domain.



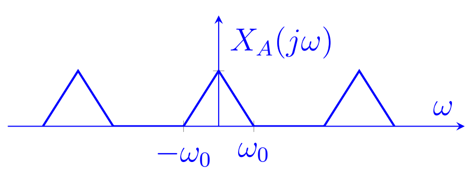

(b) Sketch $X_N(j\omega)$, the CTFT of $x_N(t) = x(t)p(t)$, when $T = T_N$.

Since $X_N(j\omega) = X(j\omega) * P(j\omega)$ and $P(j\omega)$ is an impulse train with spacing $2\omega_0$, for $X_N(j\omega)$ we should have replicates of $X(j\omega)$ centered at integer multiples of $2\omega_0$:




(c) Let $x_A(t)$ be $x(t)$ sampled at frequency $\omega_s = 2\omega_N$. Sketch $X_A(j\omega)$.

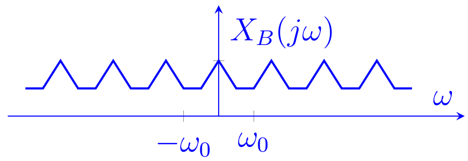
Now $P(j\omega)$ is an impulse train with spacing $4\omega_0$, so for $X_N(j\omega)$ we should have replicates of $X(j\omega)$ centered at integer multiples of $4\omega_0$:




Notice that in comparison to part (b), doubling the sampling rate (decreasing spacing between impulses in the time domain) causes the impulses to spread farther apart in the frequency domain by the same factor—in this case, 2. So, by doubling the sampling rate, you've removed every other replicate of the spectrum in the sampled signal $X_N(j\omega)$.

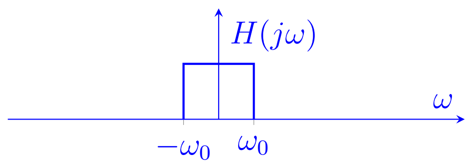
(d) Let $x_B(t)$ be $x(t)$ sampled at frequency $\omega_s = \frac{3\omega_0}{2}$. Sketch $X_B(j\omega)$. Indicate the region where aliasing occurs.

We have replicates of $X(j\omega)$ centered at integer multiples of $\frac{3\omega_0}{2}$. Note that we sum the replicates together to get the final result, so everywhere two replicates overlap and are both nonzero will result in aliasing (here, from $\frac{\omega_0}{2}$ to $\omega_0$, and every integer multiple thereof).



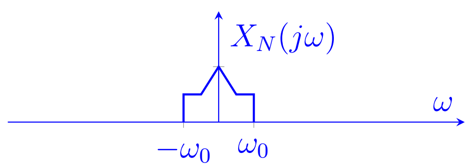

(e) Sketch $H(j\omega)$, the the CTFT of $h(t)$, the ideal reconstruction filter for a signal with the same bandwidth as $x(t)$.

The ideal reconstruction filter is a lowpass with the same bandwidth as $x(t)$:



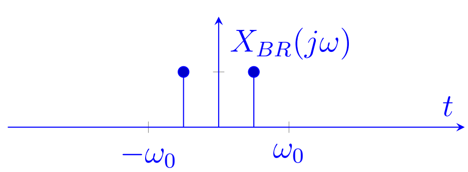

(f) Draw $X_{BR}(j\omega)$, the CTFT of $x_{BR}(t) = x_B(t) * h(t)$.

Since $X_{BR}(j\omega) = X_B(j\omega)H(j\omega)$ with $H(j\omega)$ as given above, we obtain:



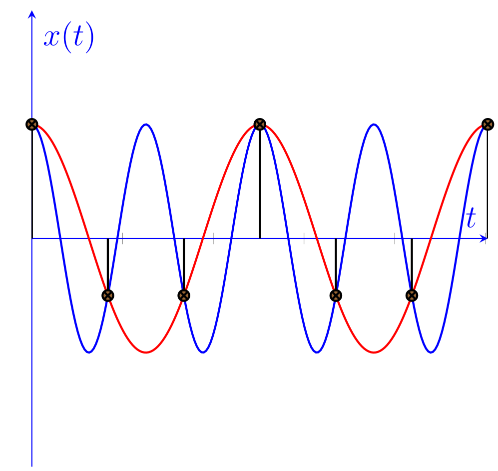

(g) Suppose that instead of the spectrum given above, we had $x(t) = \cos(\omega_0 t)$. Now what is $X_{BR}(j\omega)$?

The signal $\cos(\omega_0)t$ has highest frequency component $\omega_0$, which corresponds to the tips of the end of the triangle in the original spectrum shown at the start of this problem. Those ends of the triangle, for the replicates centered at $\pm\frac{3\omega_0}{2}$, fall at $\pm\frac{\omega_0}{2}$, as shown below:

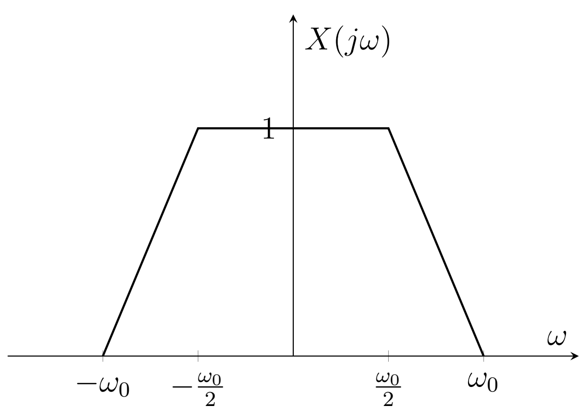



We assume that the lowpass is equal to 0 at exactly $\omega_0$ so that we do not recover the original cosine (this follows the definition of the rectangular pulse given in Lecture 6).

(h) For $x(t) = \cos(\omega_0 t)$, draw $x(t)$ and $x_{BR}(t)$ on the same plot. Circle the points where $x(t)$ was sampled to generate $x_B(t)$.



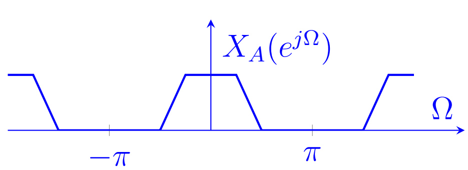

**Problem 3 (Downsampling and Upsampling)** Consider a signal $x(t)$ with the following spectrum:




Let $x_A[n]$ be the vector of samples taken from $x(t)$ with frequency $\omega_A = 4\omega_0$.

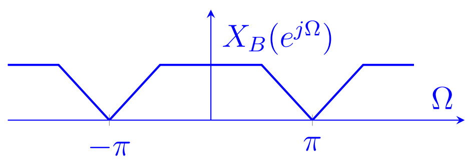
(a) Sketch $X_A(e^{j\Omega})$, the DTFT of $x_A[n]$.

Here, $\Omega$ is a normalized frequency where $\Omega = \pi$ in the DTFT corresponds to $2\omega_0$ in physical frequency (see the discussion at the start of this worksheet for more details). Hence, the DTFT appears as follows:



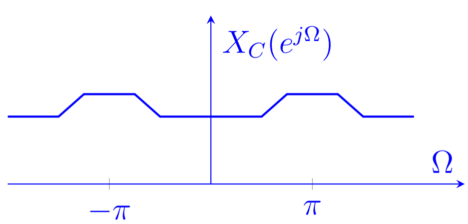
(b) Let $x_B[n] = x_A[2n]$. How would you obtain $x_B[n]$ by sampling $x(t)$, i.e., to what sampling rate $T$ does $x_B[n]$ correspond? Sketch $X_B(e^{j\Omega})$. Is there aliasing?

We have downsampled by a factor of 2 relative to the sampling rate $\omega_A$, the effect of which is to throw out every other sample from $x_A[n]$. The resulting sampling rate is $\omega_B = \frac{1}{2}\omega_A = 2\omega_0$. Thus, $T$ corresponds to $2\frac{2\pi}{4\omega_0} = \frac{\pi}{\omega_0}$. This is just at the Nyquist rate, so there is no aliasing, and indeed $\Omega = \pi$ in the DTFT corresponds to $\Omega_0$ in physical frequency.




(c) Let $x_C[n] = x_A[3n]$. What is $T$ for this sampled signal relative to $x(t)$? Sketch $X_C(e^{j\Omega})$. Is there aliasing?

We have downsampled by a factor of 3 relative to the sampling rate $\omega_A$, the effect of which is to keep only every third sample from $x_A[n]$. The resulting sampling rate is $\omega_C = \frac{1}{3}\omega_A = \frac{4\omega_0}{3}$. Hence, $T$ corresponds to $\frac{3(2\pi)}{4\omega_0} = \frac{3\pi}{2\omega_0}$, which is lower than the Nyquist rate, so there is aliasing. Note $\Omega = \pi$ corresponds to $\frac{2\omega_0}{3}$.




(d) Let $x_D[2n] = x_A[n]$ and $x_D[2n+1] = 0$. Devise a scheme to recover $x(t)$ from $x_D[n]$ *without* downsampling. Is there aliasing?

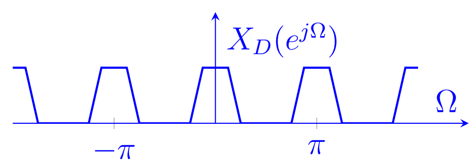
First, let's try to visualize the DTFT of $x_D[n]$. Returning to the definition of the DTFT, we see that

$$\begin{aligned} X_D\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x_D[n]e^{-j\Omega n} = \sum_{n \text{ even}} x_D[n]e^{-j\Omega n} + \sum_{n \text{ odd}} x_D[n]e^{-j\Omega n} \\ &= \sum_{n \text{ even}} x_D[n]e^{-j\Omega n} = \sum_{m=-\infty}^{\infty} x_D[2m]e^{-j\Omega(2m)} \\ &= \sum_{m=-\infty}^{\infty} x_A[m]e^{-j\Omega(2m)}, \end{aligned}$$

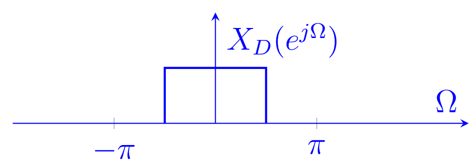
where the first step uses the fact that $x_D[2n+1] = 0$ to eliminate the sum of odd terms, and the second step uses the substitution $n = 2m$, $m = 1, 2, 3, ...$, for $n$ even. Now we define $\hat{\Omega} = 2\Omega$ and write

$$X_D\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} x_A[m]e^{-j(2\Omega)m} = \sum_{m=-\infty}^{\infty} x_A[m]e^{-j\hat{\Omega}m} = X_A\left(e^{j\hat{\Omega}}\right).$$

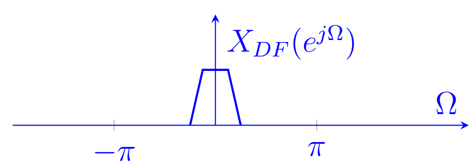
This means that $X_D(e^{j\Omega})$ corresponds to $X_A(e^{j2\Omega})$, i.e., $X_D(e^{j\Omega})$ is the DTFT of $x_A[n]$ *compressed* by a factor of 2:




If we run this through the ideal lowpass filter




then we can recover

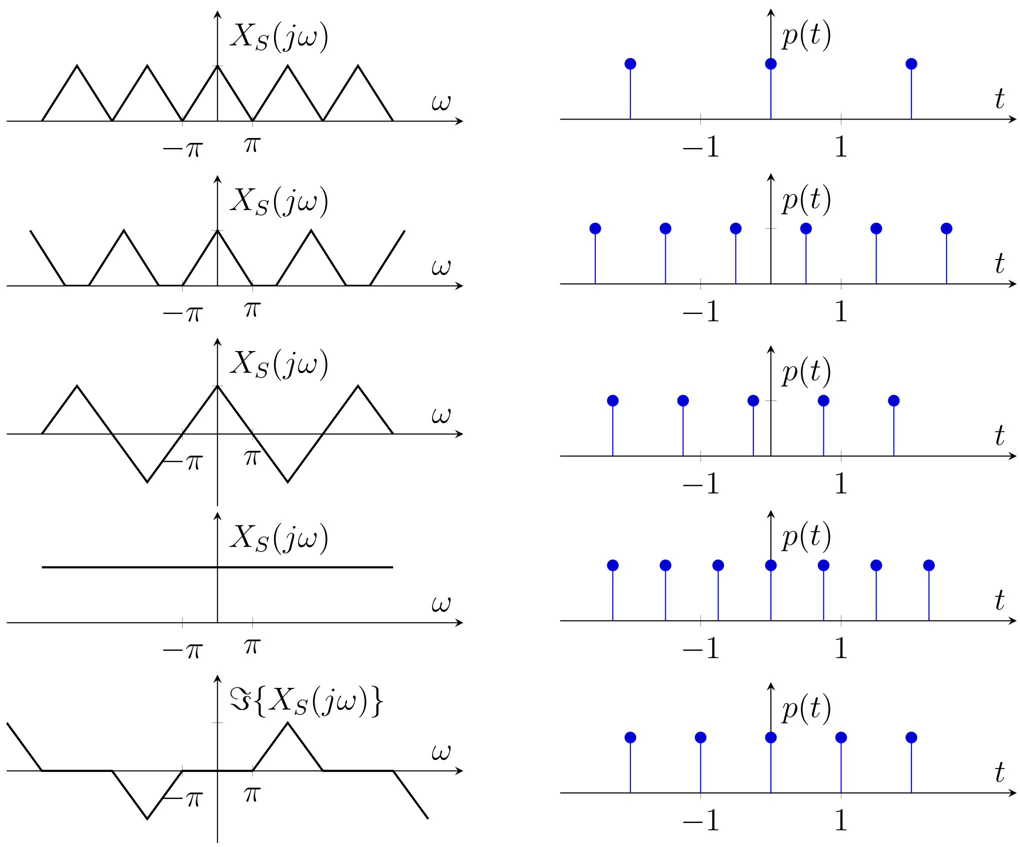



which corresponds to $x(t)$ sampled at rate $\omega_D = 8\omega_0$. (You can see this by noting that the highest physical frequency component at $\omega_0$ corresponds in the above plot to $\Omega = \frac{\pi}{4}$. Then the relation $\Omega = 2\pi\frac{\omega}{\omega_D}$ gives us $\frac{\pi}{4} = 2\pi\frac{\omega_0}{\omega_D}$, which implies $\omega_D = 8\omega_0$.) Now, since there was no aliasing in $x_A[n]$ and there is no aliasing introduced by upsampling, we can recover $x(t)$ with a sinc interpolation for $T = \frac{2\pi}{\omega_s} = \frac{2\pi}{8\omega_0} = \frac{\pi}{4\omega_0}$.

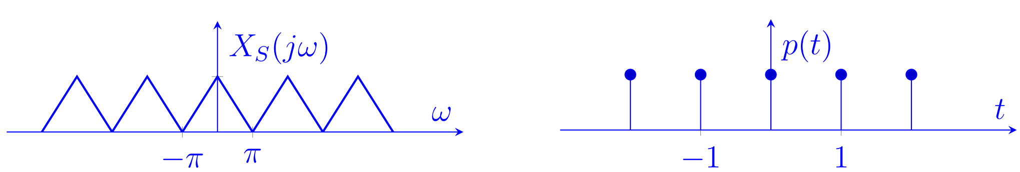
See also Lecture 14.

**Problem 4 (Uniform Sampling)** This problem was inspired by 5.10 from Osgood (2019), *Lectures on the Fourier Transform and Its Applications*.

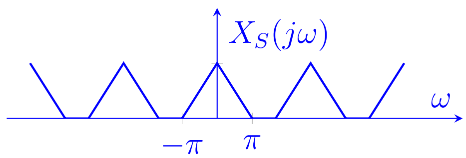
Suppose you are sampling a real signal $x(t)$ with the spectrum as given in Problem 2 above for $\omega_0 = \pi$. You take evenly spaced samples, but they are not necessarily centered at zero. Match the impulse trains $p(t)$ used for sampling to the resulting spectra of the sampled signal $x_S(t) = x(t)p(t)$.

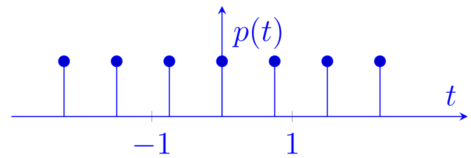



Here are the plots next to their respective sampling patterns, with corresponding explanation for the match provided below:

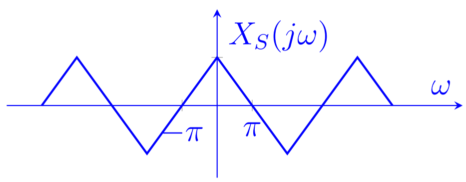


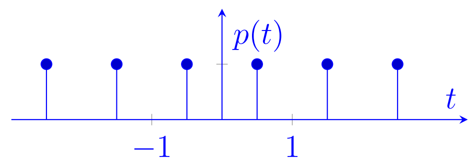

(a) The spectrum has been sampled right at the Nyquist rate, which is $\omega_N = 2\pi$. Hence, it matches the impulse train of period $T = 1$ with an impulse at $0$.



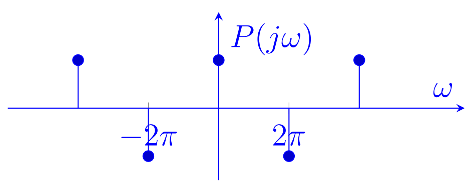




(b) This spectrum was sampled just above the Nyquist rate since there is a little gap between the ends of neighboring triangles. Hence, it matches the impulse train with a period somewhat smaller than $1$ (here, $0.75$) with an impulse centered at 0.



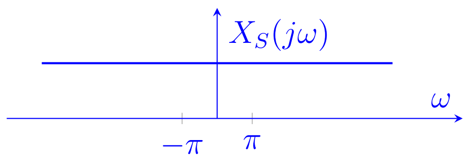

(c) This magnitude of this spectrum matches the magnitude of spectrum for uniform sampling at the Nyquist rate, so we know the impulse train that produced it has to be uniformly spaced with period $1$. Every other replicate is flipped, which indicates we somehow need the CTFT of the impulse train to look like

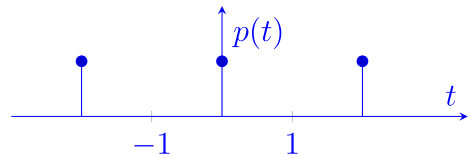



Observe that the $k$th impulse has height $(-1)^k = e^{-j\pi k} = e^{-j2\pi k\frac{1}{2}}$. Thus, for some positive constant $a$,

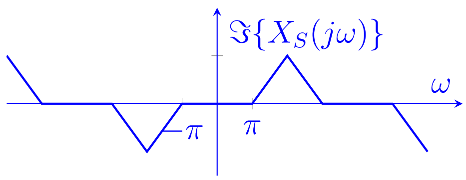
$$\begin{aligned} P(j\omega) &= a \sum_{k=-\infty}^{\infty} e^{-j2\pi k\frac{1}{2}} \delta(\omega - 2\pi k) \\ &= a \sum_{k=-\infty}^{\infty} e^{-j\omega\frac{1}{2}} \delta(\omega - 2\pi k), \end{aligned}$$

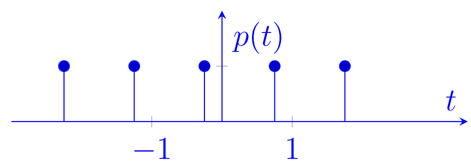
where we have used the sifting property of the $\delta$ to replace $2\pi k$ in the exponential with $\omega$. Now we recognize that the complex exponential in the frequency domain introduces a shift in the time domain, in this case of magnitude $\frac{1}{2}$. So, we must have sampled the original signal with an impulse train of period 1 shifted by $\frac{1}{2}$.






(d) This signal is severely aliased. Graphically, we can obtain it by summing replicates centered at integer multiples of $\pi$. In the time domain, this corresponds to an impulse train with period $T = 2$.






(e) Notice first that the plot represents the *imaginary* part of $X_S(j\omega)$. This means we need an impulse train with imaginary components spaced by $2\pi$ and alternating in magnitude. Based on our derivation for (c) of this same problem, we might suspect some kind of time shift is at play. Notice that $j^k$ is real for $k$ even and imaginary for $k$ odd, and furthermore that every other odd term is negative: $j^1 = j$, $j^3 = j^2 * j = -j$, and so on. Now, we know $j = e^{j\frac{\pi}{2}} = e^{j2\pi\frac{1}{4}}$, such that we can represent the frequency domain impulse train that we need as

$$\begin{aligned} P(j\omega) &= a \sum_{k=-\infty}^{\infty} e^{j2\pi k\frac{1}{4}} \delta(\omega - 2\pi k) \\ &= a \sum_{k=-\infty}^{\infty} e^{-j\omega\left(-\frac{1}{4}\right)} \delta(\omega - 2\pi k), \end{aligned}$$

which corresponds to a uniformly spaced impulse train with period 1 time shifted by $-\frac{1}{4}$.